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AUTHOR(S):

Takagi, Satoru

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# Uniqueness of Renormalized Solutions for Nonlinear Degenerate Problems

早稲田大学大学院・理工学研究科      高木 悟 (Satoru Takagi)  
Graduate School of Science and Engineering,  
Waseda University

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$ , and let  $T > 0$ . When  $N \geq 2$  we assume that  $\Omega$  has a Lipschitz boundary  $\partial\Omega$ . We consider the initial-boundary value problem

$$(E) \quad \begin{cases} \frac{\partial g(u)}{\partial t} - \Delta b(u) + \operatorname{div} \phi(u) = f & \text{in } Q = (0, T) \times \Omega, \\ b(u) = 0 & \text{on } \Sigma = (0, T) \times \partial\Omega, \\ g(u)(0, \cdot) = g(u_0) & \text{in } \Omega, \end{cases}$$

where

(H1)  $g, b : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and nondecreasing functions satisfying the normalization conditions  $g(0) = b(0) = 0$ , and  $\phi : \mathbb{R} \rightarrow \mathbb{R}^N$  is a continuous  $N$ -dimensional vector-valued function satisfying  $\phi(0) = 0$ .

(H2)  $f \in L^1(Q)$  and  $u_0 : \Omega \rightarrow \overline{\mathbb{R}}$  is measurable with  $g(u_0) \in L^1(\Omega)$ , where  $\overline{\mathbb{R}} = [-\infty, \infty]$ .

(H3) For any measurable functions  $u, v : Q \rightarrow \mathbb{R}$

$$\begin{aligned} & ((\nabla b(u) - \phi(u)) - (\nabla b(v) - \phi(v))) \cdot (\nabla u - \nabla v) \\ & + C(u, v)(1 + |\nabla b(u) - \phi(u)|^2 + |\nabla b(v) - \phi(v)|^2)|u - v| \geq 0, \end{aligned}$$

where  $C : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  is continuous.

Many authors have considered the problems like (E) as well as the stationary problems under various assumptions on the vector field and have introduced several different notions of solutions for these problems in order to prove existence and uniqueness of such solutions, see [1]-[3], [6], [10] and [14], for example.

Due to the possible degeneracy of  $b$  and  $g$ , in general, we are not able to expect that solution in the sense of distribution for (E) is unique. We thus consider the problem (E) adopting the notion of renormalized solutions. The notion of renormalized solutions was introduced by DiPerna and Lions in their papers [8] and [9] dealing with existence of a solution for the Boltzmann equation. We can also treat the problem in the case of large data in a sense by utilizing this theory. In this report we shall prove uniqueness and a comparison result of renormalized solutions for the problem (E) with no growth condition applying the method of doubling variables both in space and time introduced by Kruzhkov [12]. As to some studies of renormalized solutions, see [4], [5], [7], [11], [13], [15] and [16], for example.

We shall mention the notations and definitions. For  $k > 0$  we define a truncate function  $T_k$  by

$$T_k(u) = \begin{cases} k & \text{if } u > k \\ u & \text{if } |u| \leq k \\ -k & \text{if } u < -k \end{cases}$$

as usual. We introduce the following functions

$$S(r) = \begin{cases} 1 & \text{if } r > 0 \\ [0, 1] & \text{if } r = 0 \\ 0 & \text{if } r < 0 \end{cases}$$

and

$$S_0(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases},$$

and also define nonnegative functions  $r^+$  and  $r^-$  by  $r^+ = \max(r, 0)$  and  $r^- = -\min(r, 0)$ , respectively.

We now define a renormalized solution as in [7].

**Definition 1.1.** *A renormalized solution of (E) is a measurable function  $u : Q \rightarrow \mathbb{R}$  satisfying*

- (R1)  $g(u) \in L^1(Q)$ ,
- (R2)  $T_k(u) \in L^2(0, T; H_0^1(\Omega))$  for any  $k > 0$ ,
- (R3)  $b(T_k(u)) \in L^2(0, T; H_0^1(\Omega))$  for any  $k > 0$ ,
- (R4)  $\phi(T_k(u)) \in L^2(Q)^N$  for any  $k > 0$ ,

(R5) for all  $h \in C_0^1(\mathbb{R})$  and  $\xi \in C_0^\infty([0, T] \times \Omega)$ ,

$$\begin{aligned} \int_Q \xi_t \int_{u_0}^u h(r) dg(r) dxdt + \int_Q \xi f h(u) dxdt \\ = \int_Q (\nabla b(u) - \phi(u)) \cdot \nabla(h(u)\xi) dxdt, \end{aligned} \quad (1.1)$$

moreover,

$$\int_{Q \cap \{n \leq |u| \leq n+1\}} \nabla b(u) \cdot \nabla u dxdt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.2)$$

**Remark 1.2.** Note that each integral in (1.1) and (1.2) is well-defined. In fact, the right-hand side of (1.1) is identified with

$$\int_{Q \cap \{|u| < k\}} (\nabla b(T_k(u)) - \phi(T_k(u))) \cdot \nabla(h(T_k(u))\xi) dxdt$$

for  $k > 0$  such that  $\text{supp } h \subset (-k, k)$ . Similarly, the integral in (1.2) has to be understood as

$$\int_{Q \cap \{n \leq |u| \leq n+1\}} \nabla b(T_{n+1}(u)) \cdot \nabla T_{n+1}(u) dxdt.$$

## 2 Main theorem

We obtain the following comparison result.

**Theorem 2.1.** Suppose that (H1) and (H3) hold. Let  $u_{0i} : \Omega \rightarrow \overline{\mathbb{R}}$  be measurable with  $g(u_{0i}) \in L^1(\Omega)$ ,  $f_i \in L^1(Q)$  and let  $u_i$  be a renormalized solution of  $(E_i)$  for  $i = 1, 2$ , where

$$(E_i) \quad \begin{cases} \frac{\partial g(u_i)}{\partial t} - \Delta b(u_i) + \text{div } \phi(u_i) = f_i & \text{in } Q = (0, T) \times \Omega, \\ b(u_i) = 0 & \text{on } \Sigma = (0, T) \times \partial\Omega, \\ g(u_i)(0, \cdot) = g(u_{0i}) & \text{in } \Omega. \end{cases}$$

Then there exists  $\kappa \in S(u_1 - u_2)$  such that for a.e.  $\tau \in (0, T)$ ,

$$\begin{aligned} \int_\Omega (g(u_1)(\tau, x) - g(u_2)(\tau, x))^+ dx \\ \leq \int_\Omega (g(u_{01})(x) - g(u_{02})(x))^+ dx + \int_0^\tau \int_\Omega \kappa(f_1(t, x) - f_2(t, x)) dxdt. \end{aligned} \quad (2.1)$$

Moreover, for any  $u_0$  satisfying (H2) there exists a unique solution for (E).

In order to prove this theorem, we start with the following lemma.

**Lemma 2.2.** *Let  $u$  be a renormalized solution of (E). Then*

$$\begin{aligned} & \int_Q S_0(u - k) \left( (h(u)(\nabla b(u) - \phi(u)) + h(k)\phi(k)) \cdot \nabla \xi \right. \\ & \quad \left. - \xi f h(u) - \xi_t \int_k^u h(r) dg(r) + \xi h'(u)(\nabla b(u) - \phi(u)) \cdot \nabla u \right) dx dt \\ & \leq \int_\Omega \xi(0, x) S_0(u_0 - k) \int_k^{u_0} h(r) dg(r) dx \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} & \int_Q S_0(-k - u) \left( (h(u)(\nabla b(u) - \phi(u)) + h(-k)\phi(-k)) \cdot \nabla \xi \right. \\ & \quad \left. - \xi f h(u) - \xi_t \int_{-k}^u h(r) dg(r) + \xi h'(u)(\nabla b(u) - \phi(u)) \cdot \nabla u \right) dx dt \\ & \geq \int_\Omega \xi(0, x) S_0(-k - u_0) \int_{-k}^{u_0} h(r) dg(r) dx \end{aligned} \quad (2.3)$$

for any  $h \in C_0^1(\mathbb{R})^+$  and for any pair  $(k, \xi)$  satisfying

$$(k, \xi) \in \mathbb{R} \times C_0^\infty([0, T] \times \Omega)^+ \quad \text{or} \quad (k, \xi) \in \mathbb{R}^+ \times C_0^\infty([0, T] \times \overline{\Omega})^+, \quad (2.4)$$

where  $\mathbb{R}^+ = [0, \infty)$  and  $X^+$  denotes all nonnegative functions which belong to  $X$  with  $X = C_0^1(\mathbb{R})$ ,  $C_0^\infty([0, T] \times \Omega)$  or  $C_0^\infty([0, T] \times \overline{\Omega})$ .

**Remark 2.3.** *Note that if  $u$  is a renormalized solution of (E), then  $-u$  is a renormalized solution of the problem associated with the equation  $\tilde{g}(v)_t - \Delta \tilde{b}(v) + \operatorname{div} \tilde{\phi}(v) = \tilde{f}$ , where  $\tilde{g}(r) = -g(-r)$ ,  $\tilde{b}(r) = -b(-r)$ ,  $\tilde{\phi}(r) = -\phi(-r)$ ,  $\tilde{f} = -f$  and initial data  $\tilde{u}_0 = -u_0$ .*

*Sketch of the proof of Lemma 2.2.* Due to Remark 2.3 it is sufficient to show (2.2). Let  $h \in C_0^1(\mathbb{R})^+$ . For  $\varepsilon > 0$  let  $N_\varepsilon \in W^{1,\infty}(\mathbb{R})$  be defined by  $N_\varepsilon(r) = \inf(r^+/ \varepsilon, 1)$ . For  $\varepsilon > 0$  we see that  $N_\varepsilon(u - k)\xi \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$  for any pair  $(k, \xi)$  satisfying (2.4). Since  $u$  is a renormalized solution we find

$$\begin{aligned} G_h(u) &:= \int_0^u h(r) dg(r) \in L^1(Q), \\ \frac{\partial G_h(u)}{\partial t} &\in L^2(0, T; H^{-1}(\Omega)) + L^1(Q) \\ \text{and } G_h(u)(0, \cdot) &= \int_0^{u_0} h(r) dg(r) \in H^{-1}(\Omega) + L^1(\Omega). \end{aligned}$$

Therefore we have that

$$\begin{aligned}
& - \int_0^T \langle G_h(u)_t, N_\varepsilon(u-k)\xi \rangle dt \\
& = \int_Q \xi_t \int_{u_0}^u N_\varepsilon(r-k) dG_h(r) dx dt \\
& = \int_Q \xi_t \int_{u_0}^u N_\varepsilon(r-k) h(r) dg(r) dx dt.
\end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  on the right we obtain

$$\begin{aligned}
& \int_Q \xi_t \int_{u_0}^u S_0(r-k) h(r) dg(r) dx dt \\
& = \int_\Omega \xi(0, x) S_0(u_0 - k) \int_k^{u_0} h(r) dg(r) dx \\
& \quad + \int_Q \xi_t S_0(u-k) \int_k^u h(r) dg(r) dx dt.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& - \int_0^T \langle G_h(u)_t, N_\varepsilon(u-k)\xi \rangle dt \\
& = \int_Q (N_\varepsilon(u-k)\xi)_t \int_{u_0}^u h(r) dg(r) dx dt \\
& = - \int_Q f h(u) N_\varepsilon(u-k) \xi dx dt \\
& \quad + \int_Q (\nabla b(u) - \phi(u)) \cdot \nabla (h(u) N_\varepsilon(u-k) \xi) dx dt
\end{aligned}$$

and since  $f h(u) N_\varepsilon(u-k) \xi \in L^1(Q)$  from the Lebesgue convergence theorem it follows that

$$\lim_{\varepsilon \rightarrow 0} \left( - \int_Q f h(u) N_\varepsilon(u-k) \xi dx dt \right) = - \int_Q f h(u) S_0(u-k) \xi dx dt.$$

As to the second integral we find

$$\begin{aligned}
& \int_Q (\nabla b(u) - \phi(u)) \cdot \nabla (h(u) N_\varepsilon(u-k) \xi) dx dt \\
& = \int_Q N_\varepsilon(u-k) (\xi h'(u) (\nabla b(u) - \phi(u)) \cdot \nabla u + h(u) (\nabla b(u) - \phi(u)) \cdot \nabla \xi) dx dt \\
& \quad + \frac{1}{\varepsilon} \int_{Q \cap \{0 < u-k < \varepsilon\}} \xi h(u) (\nabla b(u) - \phi(u)) \cdot \nabla u dx dt \\
& \rightarrow \int_Q S_0(u-k) (\xi h'(u) (\nabla b(u) - \phi(u)) \cdot \nabla u + h(u) (\nabla b(u) - \phi(u)) \cdot \nabla \xi) dx dt \\
& \quad + \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{Q \cap \{0 < u-k < \varepsilon\}} \xi h(u) (\nabla b(u) - \phi(u)) \cdot \nabla u dx dt \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

Due to the divergence theorem we obtain

$$\begin{aligned}
0 &= \int_Q \operatorname{div} \left( \xi \int_0^{N_\varepsilon(u-k)} h(\varepsilon r + k) (\nabla b(\varepsilon r + k) - \phi(\varepsilon r + k)) dr \right) dx dt \\
&= \int_Q \int_0^{N_\varepsilon(u-k)} h(\varepsilon r + k) (\nabla b(\varepsilon r + k) - \phi(\varepsilon r + k)) \cdot \nabla \xi dr dx dt \\
&\quad + \frac{1}{\varepsilon} \int_{Q \cap \{0 < u - k < \varepsilon\}} \xi h(u) (\nabla b(u) - \phi(u)) \cdot \nabla u dx dt
\end{aligned}$$

whenever the pair  $(k, \xi)$  satisfies (2.4), hence

$$\begin{aligned}
&\liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{Q \cap \{0 < u - k < \varepsilon\}} \xi h(u) (\nabla b(u) - \phi(u)) \cdot \nabla u dx dt \\
&\geq \int_Q S_0(u - k) h(k) \phi(k) \cdot \nabla \xi.
\end{aligned}$$

Combining these estimates above we finally obtain (2.2).  $\square$

We next prove the following renormalized Kato inequality.

**Lemma 2.4.** *Let  $u_{0i} : \Omega \rightarrow \overline{\mathbb{R}}$  be measurable with  $g(u_{0i}) \in L^1(\Omega)$ ,  $f_i \in L^1(Q)$  and let  $u_i$  be a renormalized solution of  $(E_i)$  for  $i = 1, 2$ . Then there exists  $\kappa \in S(u_1 - u_2)$  such that for a.e.  $t \in (0, T)$ ,*

$$\begin{aligned}
& - \int_{Q \cap \{u_1 > u_2\}} \xi_t \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& - \int_{\Omega \cap \{u_{01} > u_{02}\}} \xi(0, x) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_{Q \cap \{u_1 > u_2\}} (h(u_1) (\nabla b(u_1) - \phi(u_1)) \\
& \quad - h(u_2) (\nabla b(u_2) - \phi(u_2))) \cdot \nabla \xi dx dt \\
& + \int_{Q \cap \{u_1 > u_2\}} \xi (h'(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2) (\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \\
& \leq \int_Q \xi \kappa (f_1 h(u_1) - f_2 h(u_2)) dx dt
\end{aligned} \tag{2.5}$$

for all  $h \in C_0^1(\mathbb{R})^+$  and all  $\xi \in C_0^\infty([0, T] \times \overline{\Omega})^+$ .

*Sketch of the proof of Lemma 2.4.* We adopt the method of doubling variables introduced by Kruzhkov. Thus we choose two different pairs of variables  $(s, y)$  and  $(t, x)$  in  $Q$  and consider  $u_1, f_1$  as functions in  $(s, y)$ ,  $u_2, f_2$  in  $(t, x)$ . Let  $\xi \in C_0^\infty([0, T] \times \mathbb{R}^N)^+$  be such that

$$\operatorname{supp} \xi \cap ([0, T] \times \mathbb{R}^N) \subset ([0, T] \times B)$$

where  $B$  is a ball for which

$$\begin{aligned} & \text{either } B \cap \partial\Omega = \emptyset \text{ or } B \subset\subset B' \text{ and} \\ & B' \cap \partial\Omega \text{ is a part of the graph of a Lipschitz continuous function.} \end{aligned} \quad (2.6)$$

Then there exists a sequence of mollifiers  $\sigma_l$  defined in  $\mathbb{R}$  with  $\text{supp } \sigma_l \subset (-2/l, 0)$  and there exists a sequence of mollifiers  $\rho_n$  in  $\mathbb{R}^N$  such that  $x \mapsto \rho_n(x - y) \in C_0^\infty(\Omega)$  for any  $y \in B \cap \Omega$ ,

$$\mu_n(x) = \int_{\Omega} \rho_n(x - y) dy$$

is an increasing sequence for any  $x \in B$  and  $\mu_n(x) = 1$  for any  $x \in B$  with  $d(x, \mathbb{R}^N \setminus \Omega) > c/n$ , where  $c$  is a positive constant depending on  $B$ . Further, for sufficiently large  $l$  and  $n$ , the function  $\xi^{(l,n)}$  defined by

$$\xi^{(l,n)}(t, x, s, y) = \xi(t, x) \rho_n(x - y) \sigma_l(t - s)$$

satisfies

$$\begin{aligned} (s, y) &\mapsto \xi^{(l,n)}(t, x, s, y) \in C_0^\infty([0, T] \times \overline{\Omega}) && \text{for any } (t, x) \in Q, \\ (t, x) &\mapsto \xi^{(l,n)}(t, x, s, y) \in C_0^\infty([0, T] \times \Omega) && \text{for any } (s, y) \in Q, \end{aligned}$$

and the function  $\xi^{(n)}$  defined by

$$\xi^{(n)} = \int_Q \xi^{(l,n)}(t, x, s, y) dy ds = \xi \mu_n$$

satisfies

$$\xi^{(n)} \in C_0^\infty([0, T] \times \Omega), \quad 0 \leq \xi^{(m)} \leq \xi^{(n)} \leq \xi \quad \text{for any } m \leq n.$$

We thus apply Lemma 2.2 with  $u = u_1$ ,  $k = 0$ ,  $f = f_1$ ,  $\xi = \xi^{(l,n)}(t, x, \cdot)$  and  $h(\cdot)N_\varepsilon(\cdot - u_2^+)$  in the place of  $h$ , and we have

$$\begin{aligned} & \int_Q (\xi^{(l,n)})_s \int_{u_2^+}^{u_1^+} h(r) N_\varepsilon(r - u_2^+) dg(r) dy ds \\ & + \int_{\Omega} \xi^{(l,n)}(t, x, 0, y) \int_{u_2^+}^{u_{01}^+} h(r) N_\varepsilon(r - u_2^+) dg(r) dy \\ & + \int_Q f_1 h(u_1) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)} dy ds \\ & \geq \int_Q (\nabla b(u_1) - \phi(u_1)) \cdot \nabla_y (h(u_1) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)}) dy ds \end{aligned} \quad (2.7)$$



and since  $u_2$  is a renormalized solution of  $(E_2)$  we obtain from (1.1) that

$$\begin{aligned}
& \int_Q (\xi^{(l,n)})_t \int_{u_1^+}^{u_2} h(r) N_\varepsilon(u_1^+ - r^+) dg(r) dx dt \\
& + \int_\Omega \xi^{(l,n)}(0, x, s, y) \int_{u_1^+}^{u_{02}} h(r) N_\varepsilon(u_1^+ - r^+) dg(r) dx \\
& + \int_Q f_2 h(u_2) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)} dx dt \\
& = \int_Q (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)}) dx dt. \quad (2.8)
\end{aligned}$$

Integrating (2.7) in  $(t, x)$  and (2.8) in  $(s, y)$ , respectively, over  $Q$  and taking their difference we obtain

$$\begin{aligned}
& \int_{Q \times Q} \left( (\xi^{(l,n)})_s \int_{u_2^+}^{u_1^+} h(r) N_\varepsilon(r - u_2^+) dg(r) \right. \\
& \quad \left. - (\xi^{(l,n)})_t \int_{u_1^+}^{u_2} h(r) N_\varepsilon(u_1^+ - r^+) dg(r) \right) dy ds dx dt \\
& + \left( \int_{Q \times \Omega} \xi^{(l,n)}(t, x, 0, y) \int_{u_2^+}^{u_{01}^+} h(r) N_\varepsilon(r - u_2^+) dg(r) dy dx dt \right. \\
& \quad \left. - \int_{\Omega \times Q} \xi^{(l,n)}(0, x, s, y) \int_{u_1^+}^{u_{02}} h(r) N_\varepsilon(u_1^+ - r^+) dg(r) dy ds dx \right) \\
& + \int_{Q \times Q} (f_1 h(u_1) - f_2 h(u_2)) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)} dy ds dx dt \\
& \geq \int_{Q \times Q} \left( (\nabla b(u_1) - \phi(u_1)) \cdot \nabla_y (h(u_1) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)}) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l,n)}) \right) dy ds dx dt.
\end{aligned}$$

We shall denote the three integrals on the left by  $J_1$ ,  $J_2$  and  $J_3$ , the integral on the right by  $J_4$ , respectively. We begin with the first term  $J_1$ .

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} J_1 &= \int_{Q \times Q} \left( (\xi^{(l,n)})_s S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r) dg(r) \right. \\
& \quad \left. - (\xi^{(l,n)})_t S_0(u_1^+ - u_2^+) \int_{u_1^+}^{u_2} h(r) dg(r) \right) dy ds dx dt \\
&= \int_{Q \times Q} \xi_t \rho_n \sigma_l S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r) dg(r) dy ds dx dt \\
& \quad - \int_{Q \times Q \cap \{u_1 > 0\} \cap \{u_2 < 0\}} (\xi^{(l,n)})_t \int_0^{u_2} h(r) dg(r) dy ds dx dt. \quad (2.9)
\end{aligned}$$

As to  $J_2$  we see from  $\text{supp } \sigma_l \subset (-2/l, 0)$  that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} J_2 \\
&= \int_{\Omega \times (0, 2/l) \times \Omega} \xi^{(l, n)}(0, x, s, y) S_0(u_1^+ - u_{02}^+) \int_{u_{02}}^{u_1^+} h(r) dg(r) dy ds dx \\
&= \int_{\Omega \times (0, 2/l) \times \Omega} \xi^{(l, n)}(0, x, s, y) S_0(u_1^+ - u_{02}^+) \int_{u_{02}}^{u_1^+} h(r) dg(r) dy ds dx \\
&\quad + \int_{\Omega \times (0, 2/l) \times \Omega \cap \{u_1 > 0\} \cap \{u_{02} < 0\}} \xi^{(l, n)}(0, x, s, y) \int_{u_{02}}^0 h(r) dg(r) dy ds dx. \quad (2.10)
\end{aligned}$$

In the third term we deduce that

$$\lim_{\varepsilon \rightarrow 0} J_3 = \int_{Q \times Q \cap \{u_1^+ > u_2^+\}} (f_1 h(u_1) - f_2 h(u_2)) \xi^{(l, n)} dy ds dx dt. \quad (2.11)$$

It remains to consider  $J_4$ . In terms of the divergence theorem we have

$$\begin{aligned}
J_4 &= \int_{Q \times Q} \xi^{(l, n)} N_\varepsilon(u_1 - u_2^+) h'(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 dy ds dx dt \\
&\quad + \int_{Q \times Q} h(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla_y (N_\varepsilon(u_1 - u_2^+) \xi^{(l, n)}) dy ds dx dt \\
&\quad - \int_{Q \times Q} (\nabla b(u_2^+) - \phi(u_2^+)) \cdot \nabla_x (h(u_2^+) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l, n)}) dy ds dx dt \\
&\quad + \int_{Q \times Q \cap \{u_2 < 0\}} (\nabla b(u_2^+) - \phi(u_2^+)) \cdot \nabla_x (h(u_2^+) N_\varepsilon(u_1^+ - u_2^+) \xi^{(l, n)}) dy ds dx dt \\
&\quad - \int_{Q \times Q \cap \{u_2 < 0\}} (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) N_\varepsilon(u_1^+ - u_2) \xi^{(l, n)}) dy ds dx dt \\
&= \int_{Q \times Q} \xi^{(l, n)} N_\varepsilon(u_1 - u_2^+) (h'(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
&\quad \quad \quad - h'(u_2^+) (\nabla b(u_2^+) - \phi(u_2^+)) \cdot \nabla u_2^+) dy ds dx dt \\
&\quad + \int_{Q \times Q} (h(u_1) (\nabla b(u_1) - \phi(u_1)) - h(u_2^+) (\nabla b(u_2^+) - \phi(u_2^+))) \\
&\quad \quad \quad \cdot (\nabla_x + \nabla_y) (N_\varepsilon(u_1 - u_2^+) \xi^{(l, n)}) dy ds dx dt \\
&\quad - \int_{Q \times Q \cap \{u_2 < 0\}} N_\varepsilon(u_1^+) (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) \xi^{(l, n)}) dy ds dx dt
\end{aligned}$$

Then we find

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \int_{Q \times Q} (h(u_1) (\nabla b(u_1) - \phi(u_1)) - h(u_2^+) (\nabla b(u_2^+) - \phi(u_2^+))) \\
&\quad \quad \quad \cdot (\nabla_x + \nabla_y) (N_\varepsilon(u_1 - u_2^+) \xi^{(l, n)}) dy ds dx dt \\
&\geq \int_{Q \times Q \cap \{u_1 > u_2^+\}} \rho_n \sigma_l (h(u_1) (\nabla b(u_1) - \phi(u_1)) \\
&\quad \quad \quad - h(u_2^+) (\nabla b(u_2^+) - \phi(u_2^+))) \cdot \nabla \xi dy ds dx dt \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \int_{Q \times Q} N_\varepsilon(u_1 - u_2^+) \xi^{(l,n)} (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dy ds dx dt \\
& \geq \int_{Q \times Q \cap \{u_1 > u_2^+\}} \xi^{(l,n)} (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+)) \cdot \nabla u_2) dy ds dx dt. \quad (2.13)
\end{aligned}$$

As to the remaining term we obtain from Lemma 2.2 that

$$\begin{aligned}
& \int_{Q \cap \{u_2 < 0\}} \left( (\xi^{(l,n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) + \xi^{(l,n)} f_2 h(u_2) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla (h(u_2) \xi^{(l,n)}) \right) dx dt \leq 0.
\end{aligned}$$

Since  $1 - N_\varepsilon(u_1) \geq 0$ , multiplying  $(1 - N_\varepsilon(u_1))$  to the previous inequality and integrating in  $(s, y)$  over  $Q$  we have

$$\begin{aligned}
& - \int_{Q \times Q \cap \{u_2 < 0\}} N_\varepsilon(u_1) (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) \xi^{(l,n)}) dy ds dx dt \\
& \geq \int_{Q \times Q \cap \{u_2 < 0\}} \left( (\xi^{(l,n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) + \xi^{(l,n)} f_2 h(u_2) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) \xi^{(l,n)}) \right) dy ds dx dt \\
& \quad - \int_{Q \times Q \cap \{u_2 < 0\}} N_\varepsilon(u_1) (\xi^{(l,n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) dy ds dx dt \\
& \quad - \int_{Q \times Q \cap \{u_2 < 0\}} N_\varepsilon(u_1) \xi^{(l,n)} f_2 h(u_2) dy ds dx dt \\
& \rightarrow \int_{Q \times Q \cap \{u_2 < 0\}} \left( (\xi^{(l,n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) + \xi^{(l,n)} f_2 h(u_2) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla_x (h(u_2) \xi^{(l,n)}) \right) dy ds dx dt \\
& \quad - \int_{Q \times Q \cap \{u_1 > 0\} \cap \{u_2 < 0\}} (\xi^{(l,n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) dy ds dx dt \\
& \quad - \int_{Q \times Q \cap \{u_1 > 0\} \cap \{u_2 < 0\}} \xi^{(l,n)} f_2 h(u_2) dy ds dx dt \quad \text{as } \varepsilon \rightarrow 0. \quad (2.14)
\end{aligned}$$

Combining these estimates (2.9) - (2.14) we deduce that

$$\begin{aligned}
& \int_Q \xi_t S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r) dg(r) dx dt \\
& + \int_\Omega \xi(0, x) S_0(u_{01}^+ - u_{02}^+) \int_{u_{02}^+}^{u_{01}^+} h(r) dg(r) dx \\
& + \int_Q \xi \kappa_+ S_0(u_1) (f_1 h(u_1) - (1 - S_0(-u_2)) f_2 h(u_2)) dx dt \\
& \geq \int_{Q \cap \{u_1 > u_2^+\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+))) \cdot \nabla \xi dx dt \\
& + \int_{Q \cap \{u_1 > u_2^+\}} \xi (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+)) \cdot \nabla u_2) dx dt \\
& + \lim_{n \rightarrow \infty} \int_{Q \cap \{u_2 < 0\}} \left( (\xi^{(n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) + f_2 h(u_2) \xi^{(n)} \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla (h(u_2) \xi^{(n)}) \right) dx dt
\end{aligned}$$

for any  $\xi \in C_0^\infty([0, T] \times B)^+$ , where  $\kappa_+ \in S(u_1^+ - u_2^+)$ . We also obtain from Remark 2.3 that there exists  $\kappa_- \in S(u_2^- - u_1^-)$  such that

$$\begin{aligned}
& \int_Q \xi_t S_0(u_2^- - u_1^-) \int_{-u_2^-}^{-u_1^-} h(r) dg(r) dx dt \\
& + \int_\Omega \xi(0, x) S_0(u_{02}^- - u_{01}^-) \int_{-u_{02}^-}^{-u_{01}^-} h(r) dg(r) dx \\
& + \int_Q \xi \kappa_- S_0(u_2^-) ((1 - S_0(u_1^+)) f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& \geq \int_{Q \cap \{u_2^- > u_1^-\}} (h(u_1)(\nabla b(-u_1^-) - \phi(-u_1^-)) \\
& \quad - h(u_2^-)(\nabla b(u_2^-) - \phi(u_2^-))) \cdot \nabla \xi dx dt \\
& + \int_{Q \cap \{u_2^- > u_1^-\}} \xi (h'(u_1)(\nabla b(-u_1^-) - \phi(-u_1^-)) \cdot \nabla u_1 \\
& \quad - h'(u_2^-)(\nabla b(u_2^-) - \phi(u_2^-)) \cdot \nabla u_2) dx dt \\
& - \lim_{n \rightarrow \infty} \int_{Q \cap \{u_1 > 0\}} \left( (\xi^{(n)})_t \int_{u_{01}}^{u_1} h(r) dg(r) + f_1 h(u_1) \xi^{(n)} \right. \\
& \quad \left. - (\nabla b(u_1) - \phi(u_1)) \cdot \nabla (h(u_1) \xi^{(n)}) \right) dx dt
\end{aligned}$$

for any  $\xi \in C_0^\infty([0, T] \times B)^+$ . Since  $\tilde{\kappa} = (1 - S_0(u_1^+)) S_0(-u_2^+) \kappa_- + S_0(u_1^+) \kappa_+ = (1 - S_0(-u_2)) S_0(u_1) \kappa_+ + S_0(-u_2) \kappa_- \in S(u_1 - u_2)$ , summing up the previous

two inequalities we have

$$\begin{aligned}
& \int_Q \xi_t S_0(u_1 - u_2) \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& + \int_{\Omega} \xi(0, x) S_0(u_{01} - u_{02}) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_Q \xi \tilde{\kappa}(f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2)(\nabla b(u_2) - \phi(u_2))) \cdot \nabla \xi dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} \xi (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \\
& \geq \lim_{n \rightarrow \infty} \int_{Q \cap \{u_2 < 0\}} \left( (\xi^{(n)})_t \int_{u_{02}}^{u_2} h(r) dg(r) + \xi^{(n)} f_2 h(u_2) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla (h(u_2) \xi^{(n)}) \right) dx dt \\
& - \lim_{n \rightarrow \infty} \int_{Q \cap \{u_1 > 0\}} \left( (\xi^{(n)})_t \int_{u_{01}}^{u_1} h(r) dg(r) + \xi^{(n)} f_1 h(u_1) \right. \\
& \quad \left. - (\nabla b(u_1) - \phi(u_1)) \cdot \nabla (h(u_1) \xi^{(n)}) \right) dx dt \quad (2.15)
\end{aligned}$$

for any  $\xi \in C_0^\infty([0, T) \times B)^+$ .

Now let  $\xi \in C_0^\infty([0, T) \times B)^+$ . Then  $\xi^{(m)} = \xi \mu_m \in C_0^\infty([0, T) \times \Omega)$  and we see that

$$\begin{aligned}
& - \int_{Q \cap \{u_1 > u_2\}} \xi^{(m)}_t \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& - \int_{\Omega \cap \{u_{01} > u_{02}\}} \xi^{(m)}(0, x) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_{Q \cap \{u_1 > u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) \\
& \quad - h(u_2)(\nabla b(u_2) - \phi(u_2))) \cdot \nabla \xi^{(m)} dx dt \\
& + \int_{Q \cap \{u_1 > u_2\}} \xi^{(m)} (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \\
& \leq \int_Q \xi^{(m)} \kappa(f_1 h(u_1) - f_2 h(u_2)) dx dt.
\end{aligned}$$

nce  $\xi^{(m)} = \xi - \xi(1 - \mu_m)$  we obtain from (2.15) that

$$\begin{aligned}
& \int_{Q \cap \{u_1 > u_2\}} \xi_t \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& + \int_{\Omega \cap \{u_{01} > u_{02}\}} \xi(0, x) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_Q \xi \tilde{\kappa} (f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& + \int_Q \xi (\kappa - \tilde{\kappa}) (f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (h(u_1) (\nabla b(u_1) - \phi(u_1)) \\
& \quad - h(u_2) (\nabla b(u_2) - \phi(u_2))) \cdot \nabla \xi dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} \xi (h'(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2) (\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \\
& \geq \int_{Q \cap \{u_1 > u_2\}} (\xi(1 - \mu_m))_t \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& + \int_{\Omega \cap \{u_{01} > u_{02}\}} (\xi(1 - \mu_m))(0, x) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_Q (\xi(1 - \mu_m)) \tilde{\kappa} (f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& + \int_Q (\xi(1 - \mu_m)) (\kappa - \tilde{\kappa}) (f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (h(u_1) (\nabla b(u_1) - \phi(u_1)) \\
& \quad - h(u_2) (\nabla b(u_2) - \phi(u_2))) \cdot \nabla (\xi(1 - \mu_m)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (\xi(1 - \mu_m)) (h'(u_1) (\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2) (\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \\
& \geq \lim_{n \rightarrow \infty} \int_{Q \cap \{u_2 < 0\}} \left( (\xi(1 - \mu_m) \mu_n)_t \int_{u_{02}}^{u_2} h(r) dg(r) + (\xi(1 - \mu_m) \mu_n) f_2 h(u_2) \right. \\
& \quad \left. - (\nabla b(u_2) - \phi(u_2)) \cdot \nabla (h(u_2) \xi(1 - \mu_m) \mu_n) \right) dx dt \\
& - \lim_{n \rightarrow \infty} \int_{Q \cap \{u_1 > 0\}} \left( (\xi(1 - \mu_m) \mu_n)_t \int_{u_{01}}^{u_1} h(r) dg(r) + (\xi(1 - \mu_m) \mu_n) f_1 h(u_1) \right. \\
& \quad \left. - (\nabla b(u_1) - \phi(u_1)) \cdot \nabla (h(u_1) \xi(1 - \mu_m) \mu_n) \right) dx dt \\
& + \int_{\cap} (\xi(1 - \mu_m)) (\kappa - \tilde{\kappa}) (f_1 h(u_1) - f_2 h(u_2)) dx dt.
\end{aligned}$$

In the last term on the right it is clear that the integral converges to 0 as  $m \rightarrow \infty$ . Since if  $n$  is large enough then  $\mu_n = 1$  on  $\text{supp } \mu_m$  we find that  $(1 - \mu_m)\mu_n = \mu_n - \mu_m$ . Therefore the remaining terms tend to 0 as  $m \rightarrow \infty$ . It implies that

$$\begin{aligned}
& \int_Q \xi_t S_0(u_1 - u_2) \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& + \int_{\Omega} \xi(0, x) S_0(u_{01} - u_{02}) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_Q \xi \kappa(f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2)(\nabla b(u_2) - \phi(u_2))) \cdot \nabla \xi dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} \xi (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \geq 0,
\end{aligned}$$

with  $\kappa \in S(u_1 - u_2)$ .

To this end, let  $B_0 \subset\subset \Omega$  be such that  $\cup_{i=0}^n B_i$  is a covering of  $\Omega$ , where  $B_i$  is a ball satisfying (2.6) for  $i = 1, \dots, n$ . Let  $\{\nu_i\}_{i=0}^n$  be such that  $\nu_i \in C_0^\infty(B_i)$  for  $i = 0, 1, \dots, n$  and let  $\xi \in C_0^\infty([0, T] \times \overline{\Omega})^+$ . Then for  $i = 0, 1, \dots, n$  we have

$$\begin{aligned}
& \int_Q (\xi \nu_i)_t S_0(u_1 - u_2) \int_{u_2}^{u_1} h(r) dg(r) dx dt \\
& + \int_{\Omega} (\xi \nu_i)(0, x) S_0(u_{01} - u_{02}) \int_{u_{02}}^{u_{01}} h(r) dg(r) dx \\
& + \int_Q (\xi \nu_i) \kappa(f_1 h(u_1) - f_2 h(u_2)) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2)(\nabla b(u_2) - \phi(u_2))) \cdot \nabla (\xi \nu_i) dx dt \\
& - \int_{Q \cap \{u_1 > u_2\}} (\xi \nu_i) (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\
& \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dx dt \geq 0.
\end{aligned}$$

Since  $\xi = \sum_{i=0}^n (\xi \nu_i)$  we obtain (2.5) for any  $\xi \in C_0^\infty([0, T] \times \overline{\Omega})^+$ .  $\square$

### 3 Proof of the main theorem

We finally give the proof of our main result.

*Proof of Theorem 2.1.* Let  $u_i$  be a renormalized solution of  $(E_i)$  for  $i = 1, 2$ . Choosing  $\xi = \alpha \otimes 1$  with  $\alpha \in C_0^\infty([0, T])$  in (2.5) there exists  $\kappa \in S(u_1 - u_2)$

$$\begin{aligned} & - \int_Q \alpha_t \left( S_0(u_1 - u_2) \int_{u_2}^{u_1} h(r) dg(r) - S_0(u_{01} - u_{02}) \int_{u_{02}}^{u_{01}} h(r) dg(r) \right) dxdt \\ & \quad + \int_{Q \cap \{u_1 > u_2\}} \alpha (h'(u_1)(\nabla b(u_1) - \phi(u_1)) \cdot \nabla u_1 \\ & \quad \quad \quad - h'(u_2)(\nabla b(u_2) - \phi(u_2)) \cdot \nabla u_2) dxdt \\ & \leq \int_Q \alpha \kappa (f_1 h(u_1) - f_2 h(u_2)) dxdt \end{aligned} \quad (3.1)$$

for any  $h \in W^{1,\infty}(\mathbb{R})$  with compact support. We now define the function  $h_n \in W^{1,\infty}(\mathbb{R})$  by  $h_n(r) = \inf((n+1 - |r|)^+, 1)$  and replace  $h$  by  $h_n$  in (3.1). As to the second integral on the left we divide as

$$\begin{aligned} & \int_{Q \cap \{u_1 > u_2\}} \alpha h_n'(u_1) \nabla b(u_1) \cdot \nabla u_1 dxdt - \int_{Q \cap \{u_1 > u_2\}} \alpha h_n'(u_2) \nabla b(u_2) \cdot \nabla u_2 dxdt \\ & \quad - \int_{Q \cap \{u_1 > u_2\}} \alpha (h_n'(u_1) \phi(u_1) \cdot \nabla u_1 - h_n'(u_2) \phi(u_2) \cdot \nabla u_2) dxdt. \end{aligned}$$

Since  $u_1, u_2$  are renormalized solutions we see from (1.2) that the first two integrals on the right tend to 0 as  $n \rightarrow \infty$ . Moreover, thanks to the divergence theorem we have

$$\begin{aligned} & - \int_{Q \cap \{u_1 > u_2\}} \alpha (h_n'(u_1) \phi(u_1) \cdot \nabla u_1 - h_n'(u_2) \phi(u_2) \cdot \nabla u_2) dxdt \\ & \quad = \int_Q \alpha \operatorname{div} \left( - \int_{\inf(u_1, u_2)}^{u_1} h_n'(r) \phi(r) dr \right) dxdt = 0. \end{aligned}$$

Therefore the second integral on the left in (3.1) converges to 0 as  $h = h_n \rightarrow 1$  and it implies that

$$\begin{aligned} & - \int_Q \alpha_t ((g(u_1)(t, x) - g(u_2)(t, x))^+ - (g(u_{01})(x) - g(u_{02})(x))^+) dxdt \\ & \leq \int_Q \alpha \kappa (f_1(t, x) - f_2(t, x)) dxdt \end{aligned}$$

for all  $\alpha \in C_0^\infty([0, T])$ . We thus conclude the proof of our main theorem.  $\square$



## References

- [1] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Zeitschrift **183** (1983) 311-341.
- [2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vazquez, *An  $L^1$ -theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (1995) 241-273.
- [3] P. Bénilan and P. Wittbold, *On mild and weak solutions of elliptic-parabolic problems*, Adv. Differential Equations **1** (1996) 1053-1073.
- [4] D. Blanchard and H. Redwane, *Renormalized solutions for a class of nonlinear evolutionary problems*, J. Math. Pures Appl. **77** (1998) 117-151.
- [5] L. Boccardo, D. Giachetti, J. I. Diaz and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms*, J. Differential Equations **106** (1993) 215-237.
- [6] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Ration. Mech. Anal. **147** (1999) 269-361.
- [7] J. Carrillo and P. Wittbold, *Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems*, J. Differential Equations **156** (1999) 93-121.
- [8] R. J. DiPerna and P.-L. Lions, *Global existence for the Fokker-Plank-Boltzmann equations*, Comm. Pure Appl. Math. **11** No.2 (1989) 729-758.
- [9] R. J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Ann. of Math. **130** (1989) 321-366.
- [10] K. Kobayasi, *The equivalence of weak solutions and entropy solutions of nonlinear degenerate second order equations*, to appear.
- [11] K. Kobayasi, S. Takagi and T. Uehara, *Uniqueness of renormalized solutions of degenerate quasilinear elliptic equations*, Gakujutsu Kenkyu, Series of Math., School of Education, Waseda Univ. **49** (2001) 5-15.
- [12] S. N. Kruzhkov, *First-order quasilinear equations in several independent variables*, Math. USSR Sb. **10** (1970) 217-243.
- [13] G. D. Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999) 741-808.
- [14] F. Otto,  *$L^1$ -contraction and uniqueness for quasilinear elliptic-parabolic equations*, J. Differential Equations **131** (1996) 20-38.

- [15] J. M. Rakotoson, *Uniqueness of renormalized solutions in a  $T$ -set for the  $L^1$ -data problem and the link between various formulations*, Indiana Univ. Math. J. **43** (1994) 285-293.
- [16] S. Takagi, *Uniqueness of renormalized solutions for nonlinear degenerate second order equations*, in preparation.